Rate-Based Stochastic Fusion Calculus and Continuous Time Markov Chains*

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1 Introduction

This paper presents a stochastic fusion calculus suitable to describe systems involving general patterns of interactions. We start from fusion calculus [8] which is a symmetric generalisation of the π-calculus, and present a rate-based stochastic fusion calculus, providing a concise and compositional way to describe the behaviour of complex systems by using probability distributions.

We provide the semantics of stochastic fusion calculus by using rate-based transition systems [4] in the elegant and general variant proposed by De Nicola et al. [2]. The stochastic nature of the new transition systems is given by the fact that transition labels represent actions, and the transition result is a function associating a positive real value to each possible target process, expressing the stochastic rate of an exponential distribution modelling the duration of the transition. For two processes running in parallel, we define the distribution of their synchronisation using their apparent rates. Associativity of parallel composition is a particularly desirable property either in the context of network and distributed systems, either in the context of biological systems, where parallel composition is often used to model molecular populations. Following the approach proposed in [2], associativity of the parallel composition operator is guaranteed in the rate-based stochastic semantics of the fusion calculus (differently to what happens in the stochastic π-calculus [9] and in a previous formalisation of a stochastic fusion calculus [1]).

We extend the notion of hyperbisimulation to stochastic fusion calculus, and prove that the stochastic hyperequivalence is a congruence. The rate-based transition system resulting from a stochastic fusion process leads the expression of a continuous time Markov chain which preserves the notion of hyperequivalence.

The modelling power of the stochastic fusion calculus is suggested by an example where we formalise some of the one-to-many interactions occurring between a plant root and a particular kind of fungi in the arbuscular mycorrhizal symbiosis. A quantitative simulation is performed using the PRISM model checker on the continuous time Markov chain extracted from the rate-based transition system describing such interactions by means of stochastic fusion processes.

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2 Rate-Based Stochastic Fusion Calculus

The fusion calculus was introduced by Parrow and Victor as a symmetric generalisation of the π-calculus [8]. The π-calculus has two binding operators (prefix and restriction), the effects of communication are local, and input and output actions are asymmetric. Unlike the π-calculus, the fusion calculus has only one binding operator, and the effects of communication are both local and global. Fusion calculus makes input and output operations fully symmetric: a more appropriate terminology for them might be action and co-action. A fusion is a name equivalence which allows to use interchangeably all the names of an equivalence class in a term of the calculus. Computationally, a fusion is generated as the result of a synchronisation between two complementary actions, and it is propagated to processes running in parallel within the same scope of the fusion. In practice, the effect of a fusion could be seen as the update of a (global) shared state. Fusions are suitable to express general patterns of interactions, including one-to-many and many-to-many interactions. We remind to [8] for details about the syntax and semantics of the fusion calculus.

Many phenomena which take place in practice are described by non-exponential distributions, and stochastic fusion calculus could be defined by using general distributions. For the sake of simplicity, we use here the exponential distribution, inheriting some properties derived from the memoryless feature of this distribution: the time at which a state change occurs is independent of the time at which the last state change occurred. In this way we do not have to keep track of the past state transitions (e.g. in an implementation).

Following the variant of rate transition systems [4] introduced in [2], we define the semantics of SFC via a transition relation $P \xrightarrow{\delta} \rho$ associating to a given process P and a transition action label δ a *next state function* (NSF) $\rho : \mathcal{SFC} \to \mathbb{R}^{\geq 0}$.

3 Stochastic Hyperbisimulation

Several papers of the last two decades define Markovian bisimulations, we mention the seminal paper by Larsen and Skou [7]. The definition of stochastic hyperbisimulation is also related to the notion of lumpability for Markov chains [5] (also see the next section). Two processes P and Q are lumping equivalent, and we denote this by $P \sim Q$, if the total rate of moving to an equivalence class S under $\sim$ is identical for both processes. Lumping equivalence also preserves stochastic rewards while reducing the size of the underlying stochastic transition system.

Two processes P and Q are stochastic bisimilar, written $P \dot{\sim}_{SH} Q$, if they are related by a stochastic bisimulation. Stochastic bisimilarity is not a congruence in the fusion calculus. We therefore look for the largest congruence included in the stochastic bisimilarity. This is achieved by closing the definition of stochastic bisimulation under arbitrary substitutions.

Definition 1 (Stochastic Hyperbisimulation). *A stochastic hyperbisimulation is an equivalence relation $\mathcal{R}$ over $\mathcal{SFC}$ satisfying the following properties:*

i) *$\mathcal{R}$ is closed under any substitution σ, i.e., $P\mathcal{R}Q$ implies $P\sigma\mathcal{R}Q\sigma$ for any σ;*
ii) *for each pair $(P,Q) \in \mathcal{R}$, for all actions δ, and for all equivalence classes $S \in \mathcal{SFC}/\mathcal{R}$, we have $\gamma_\delta(P,S) = \gamma_\delta(Q,S)$.*

Two processes P and Q are stochastic hyperbisimulation equivalent (or stochastic hyperequivalent) if they are related by a stochastic hyperbisimulation. We write $P \sim_{SH} Q$.

The following holds.

Theorem 1. *(**Congruence**) Stochastic hyperequivalence is a congruence, i.e., for $P,Q \in \mathcal{SFC}$ and $C \in \mathcal{SFC}[\,]$, $P \sim_{SH} Q$ implies $C[P] \sim_{SH} C[Q]$.*

4 Stochastic Fusion Processes as CTMCs

We provide a mechanism to translate the rate-based transition system deriving from the stochastic semantics into a Continuous Time Markov Chain (CTMC), providing a wide set of means for automatic verification.

By construction, the following holds.

Theorem 2. *Stochastic hyperequivalence preserves strong Markovian bisimulation, i.e., for $P,Q \in \mathcal{SFC}$, $P \sim_{SH} Q$ implies that the related CTMCs are Markovian bisimilar.*

5 Modelling the Arbuscular Mycorrhizal Symbiosis

The arbuscular mycorrhizal (AM) symbiosis is an example of association with high compatibility formed between fungi belonging to the Glomeromycota phylum and the roots of most land plants [3]. AM fungi are obligate symbionts, in the absence of a host plant, spores of AM fungi germinate and produce a limited amount of mycelium. The recognition between the two symbionts is driven by the perception of diffusible signals and, once reached the root surface, the AM fungus enters in the root, overcomes the epidermal layer and it grows inter-and intracellularly all along the root in order to spread fungal structures. Once inside the inner layers of the cortical cells the differentiation of specialised, highly branched intracellular hyphae called arbuscules occur. Arbuscules are considered the major site for nutrients exchange between the two organisms. The fungus supply the host with essential nutrients such as phosphate, nitrate and other minerals from the soil. In return, AM fungi receive carbohydrates derived from photosynthesis in the host. AM symbiosis also confers resistance to the plant against pathogens and environmental stresses. The colonisation of the host plant requires the accomplishment of two main events: i) signalling and partner recognition, ii) the colonisation of root tissues and the development of intraradical fungal structures leading to a functional symbiosis.

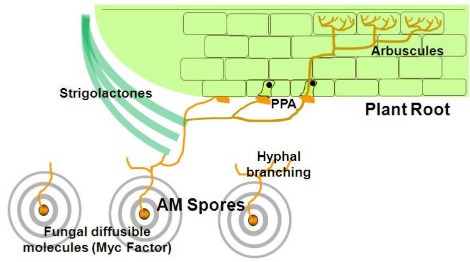

Fig. 1. The Arbuscular Mycorrhizal symbiosis

The interaction begins with a molecular dialogue between the plant and the fungus [3]. Host roots release signalling molecules characterised as *strigolactones*. Within just a few hours, strigolactones at subnanomolar concentrations induce alterations in fungal physiology, mitochondrial activity and extensive hyphal branching (leading the fungal spore to produce hyphae towards the plant root).

The external signal released by AM fungi (called *myc factor*) is perceived by a receptor on the plant plasma membrane and is transduced into the cell with the activation of a symbiotic signalling pathway that lead to the colonisation process (Pre Penetration Apparatus, PPA).

We model these initial communication between the plant root and AM fungi with SFC and simulated the resulting CTMC with the PRISM model checker.

An extended report about the work on this paper is available at: `http://www.di.unito.it/~troina/ictcs12/stocFusion.pdf`.

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